International Journal of Engineering, Science and Mathematics
Vol. 10 Issue 09, September 2021,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

Premeasure Spaces, Tight Functions And Extension toQuasi*-Measure

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Abstract

The present paper deals with the theory of [0, 1] valued maps defined on a nonempty set X. We have concentrated over the study of two types of functions, viz. tight functions and smooth functions. The notions of lower and upper envelopes of a function β defined on a sublattice K of I^X are introduced, and are extensively used to prove several results. Finally it is obtained that every supermodular and smooth from above function can be extended to an inner regular quasi*-measure.

Introduction

In measure theory, a basic procedure is that of extending the notion of a "measure" on a given class of sets to a larger class of sets. Kelley, Nayak and Srinivasan [**4**] proved that a nonnegative real valued function μ defined on a lattice A of sets is a premeasure (meaning that it extends to a countably additive measure on a δ-ring of sets containing A) provided μ is tight and continuous at ϕ. The extension of this theorem to the case of a real valued (not necessarily nonnegative real valued) function is dealt in [**9**]. In 1981, Morales [**8**] established a quite general extension theorem for a uniform semigroup-valued tight set function λ on a lattice L, the domain of extension being the σ-ring generated by L. He also discussed the extension of λ on the σ-algebra of locally L-measurable sets. The problem of generation of measures by tight functions defined on a lattice of sets has been taken up by several workers [**2, 6, 10, 11, 12**]. Adamski [**1**] proved that every nonnegative, semifinite, smooth at ϕ, tight function defined on a lattice of sets can be extended to an inner regular measure. Besides proving results on the approximation of measurable sets by members of a lattice A, Kelley and Srinivasan [**5**] proved that every function $\mu : A \rightarrow R^+$, which is tight and smooth from above at ϕ is a premeasure (here A is closed under countable intersections). In [**9**], a weaker condition for tightness than in [**5**] is used, aiming at its adaptation to the vector valued case. Recently we have obtained a Jordan decomposition type theorem for a weakly tight function under suitable conditions [**7**].

In Section 2 of this paper we have proved a theorem on characterization of a modular [0, 1] valued function β defined on a lattice K of elements in I^X. The notions of lower and upper envelopes of β, introduced in this section, are extensively used in the rest of the paper. These notions lead to the definition of a tight function, a particular case of a ρ-tight function which turns out to be monotone and modular.

Section 3 deals with the study of functions which are smooth from above and we also obtain that if $\beta : K \to I$ is smooth from above, then for any $f \in K_\delta$, its upper envelope $\beta^*(f)$ can be expressed as the limit of the sequence $\{\beta(f_n)\}_{n=1}^{\infty}$, where $\{f_n\}_{n=1}^{\infty}$ is a sequence in K decreasing to f. While giving the notion of a $\hat{\beta}$-function with the help of upper envelopes, we observe that $\hat{\beta}$ is K_δ-inner regular. Finally, it is proved that every supermodular and smooth from above function defined on K can be extended to a K_δ-inner regular quasi*-measure.

Notations. Throughout this paper, X denotes a nonempty set and $I \equiv [0, 1]$ is the closed unit interval of the real line R; C denotes a subfamily of I^X of all functions from X to I; K stands for a sublattice of I^X containing the least element **0** and the greatest element **1,** where **0** and **1** are constant functions sending each $x \in X$ to 0 and 1 respectively; $\beta : K \to I$ and $\rho : I^X \to I$ denote functions satisfying $\beta(\mathbf{0}) = 0$ and $\rho(\mathbf{0}) = 0$. We call the triple (X, K, β) a premeasure space. The family of all countable meets of elements in K is denoted by K_δ.

2. Measuring Envelopes and Tight Functions

Let $C \subseteq I^X$ be a lattice and $\xi : C \to I$ be a function. We call ξ *monotone* if, $f, g \in C$, $g \le f \Rightarrow \xi(g) \le \xi(f)$. The function ξ is to be called *modular* if $\xi(f) + \xi(g) = \xi(f \vee g) + \xi(f \wedge g)$, $f, g \in C$. A function $\beta : K \to I$ is called *semifinite* if, for every $f \in K$, $\beta(f) = \sup\{\beta(g) : g \le f, g \in K\}$.

Proposition 2.1. Let (X, K, β) be a premeasure space. The function β is modular if and only if

$$\beta(f_1)+\beta(g_1)=\beta(f_2)+\beta(g_2), \qquad 2.1.1$$

where $f_1, f_2, g_1, g_2 \in$ K with $f_1 \wedge g_1 = f_2 \wedge g_2$ and $f_1 \vee g_1 = f_2 \vee g_2$.

Proof. Let β be modular. For $f_1, f_2, g_1, g_2 \in K$ such that $f_1 \wedge g_1 = f_2 \wedge g_2$ and $f_1 \vee g_1 = f_2 \vee g_2$,we have $\beta(f_1 \vee g_1)=\beta(f_2 \vee g_2)$, and so $\beta(f_1)+\beta(g_1)-\beta(f_1 \wedge g_1)=\beta(f_2)+\beta(g_2)-\beta(f_2 \wedge g_2)$.Since $f_1 \wedge g_1 = f_2 \wedge g_2$, we get $\beta(f_1)+\beta(g_1)=\beta(f_2)+\beta(g_2)$.

Conversely, let (2.1.1) hold. Let $f, g \in$ K. Since $((f \vee g) \vee (f \wedge g)) = f \vee g$ and $((f \vee g) \wedge (f \wedge g)) = f \wedge g$, (2.1.1) yields $\beta(f)+\beta(g)=\beta(f \vee g)+\beta(f \wedge g)$, i.e. β is modular.

Proposition 2.2. Let β : K → I be a function. Suppose that $f_1, f_2, g_1, g_2 \in K$, $f_1 \le f_2, g_1 \le g_2$ and $f_2 - f_1 = g_2 - g_1$. If β satisfies $\beta(f_2)-\beta(f_1)=\beta(g_2)-\beta(g_1)$, *then* β *is modular.*

Proof. Let $f, g \in$ K. Since $(f \vee g) - f = g - (f \wedge g)$, we get $\beta(f \vee g)-\beta(f)$ $=\beta(g)-\beta(f \wedge g)$, or $\beta(f \vee g)+\beta(f \wedge g)=\beta(f)+\beta(g)$.

Definitions 2.3. We define $\beta_* : I^X \to I$ and

$\beta_*(f) = \sup\{\beta(g) : g \le f, g \in K\}$

and

$\beta^*(f) = \inf\{\beta(g) : f \le g, g \in K\}$,

for $f \in I^X$ and call β_* and β^* the *lower envelope* and the *upper envelope of* β respectively.

We obtain:

(i) $\beta^*(\mathbf{0}) = 0 = \beta_*(\mathbf{0})$;

(ii) both β_* and β^* are monotone;

(iii) $\beta^* | K \le \beta \le \beta_* | K$;

(iv) β is semifinite iff β is monotone iff $\beta^* | K = \beta = \beta_* | K$.

Definitions 2.4. Let $\beta : K \to I$ and $\rho : I^X \to I$ with $\beta(\mathbf{0}) = 0$ and $\rho(\mathbf{0}) = 0$. Then β is called *ρ-tight* if

$$\beta(f_2) = \beta(f_1) + \rho(f_2 - f_1),\ f_1, f_2 \in K,\ f_1 \le f_2.$$

The function β is called *tight* if β is β_*-tight.

Proposition 2.5. *Let β be ρ-tight. Then*

(i) *ρ is an extension of β;*

(ii) *β is monotone;*

(iii) *β is modular.*

Proof. We shall proved only (iii).

(iii) Let $f_1, f_2 \in K$. Since $(f_1 \vee f_2) - f_2 = f_1 - (f_1 \wedge f_2)$, we get $\beta(f_1 \vee f_2) - \beta(f_2) = \rho(f_1 \vee f_2 - f_2) = \rho(f_1 - f_1 \wedge f_2) = \beta(f_1) - \beta(f_1 \wedge f_2)$.

Definitions 2.6. Let $\rho : I^X \to I$ be a function with $\rho(\mathbf{0}) = 0$. We call $f \in I^X$ *ρ-measurable* if

$$\rho(g) = \rho(g \wedge f) + \rho(g - g \wedge f)$$

for all g in I^X. The family of all ρ-measurable functions is denoted by $M(\rho)$. We define $M(\rho; K) = \{f \in I^X : \rho(g) = \rho(g \wedge f) + \rho(g - g \wedge f) \text{ for all } g \in K\}$;

and, for $D \subseteq I^X$, $F(D) = \{f \in I^X : f \wedge g \in D, \text{ for all } g \in D\}$.

We obtain :

(i) The functions $\mathbf{0}$ and $\mathbf{1}$ are ρ-measurable.

(ii) If f is ρ-measurable, then $\rho(f') = \rho(\mathbf{1}) - \rho(f)$, for $\rho(\mathbf{1}) = \rho(\mathbf{1} \wedge f) + \rho(\mathbf{1} - \mathbf{1} \wedge f)$. Here $f' = \mathbf{1} - f$.

Proposition 2.7. *If $K \subseteq M(\rho; K)$ and ρ is an extension of β, then β is ρ-tight.*

Proof. Let $f_1, f_2 \in K$ with $f_1 \le f_2$. Then $f_1, f_2 \in M(\rho; K)$ and so, for any $g \in K$, $\rho(g) = \rho(g \wedge f_i) + \rho(g - g \wedge f_i),\ i = 1, 2$. Consequently,

$\beta(f_2) = \rho(f_2) = \rho(f_1 \wedge f_2) + \rho(f_2 - f_1 \wedge f_2) \quad = \beta(f_1) + \rho(f_2 - f_1)$, showing that β is ρ-tight.

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Proposition 2.8. *Let* $\rho : I^X \to I$ *satisfy* $\rho(\mathbf{0}) = 0$. *Let* β *be* ρ*-tight. Then*

(i) $K \subseteq \mathrm{M}(\rho; K)$;

(ii) *for any* $D \subseteq I^X$ *with* $K \subseteq D \subseteq M(\rho; K)$, $\mathrm{F(D)} \subseteq \mathrm{M}(\rho; K)$.

Proof. (i) Since β is ρ-tight, by Proposition 2.5, ρ is an extension of β. Let $f \in K$. Then, for any $g \in K$, $\rho(g) = \beta(g) = \beta(g \wedge f) + \rho(g - g \wedge f) = \rho(g \wedge f) + \rho(g - g \wedge f)$. Hence $K \subseteq \mathrm{M}(\rho; K)$.

(ii) Let $g \in \mathrm{F(D)}$. Then $f \wedge g \in \mathrm{D} \subseteq \mathrm{M}(\rho; K)$, for each $f \in \mathrm{D}$. Let $h \in K$. Then $h \in \mathrm{D}$, and so $h \wedge g \in \mathrm{M}(\rho; K)$. Also

$$\rho(h) = \rho(h \wedge h \wedge g) + \rho(h - h \wedge h \wedge g) = \rho(h \wedge g) + \rho(h - h \wedge g),$$

which yields that $g \in \mathrm{M}(\rho; K)$.

Theorem 2.9 . *Let* β *be monotone. Then the following statements are equivalent*:

(i) β *is tight*.

(ii) $K \subseteq \mathrm{M}(\beta_*; K)$.

(iii) $\mathrm{F}(K) \subseteq \mathrm{M}(\beta_*; K)$.

Proof. Since β is monotone, by (2.3) (iv), $\beta_* \mid K = \beta$.

(i) $\Rightarrow$ (ii). Follows from Proposition 2.8 (i).

(ii) $\Rightarrow$ (i). Follows from Proposition 2.7.

(ii) $\Rightarrow$ (iii). Let $g \in \mathrm{F}(K)$. Then, for each $h \in K$, $h \wedge g \in K \subseteq M(\beta_*; K)$.

Hence, for $h \in K$, $\beta_*(h \wedge g) + \beta_*(h - h \wedge g) = \beta_*(h)$, i.e. $g \in \mathrm{M}(\beta_*; K)$.

Finally, since $K \subseteq \mathrm{F}(K)$, (iii) $\Rightarrow$ (ii) holds.

Similarly, we obtain the following :

Proposition 2.10. *Let* β *be monotone. Then* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), *where*

(i) $\mathrm{F}(K) \subseteq \mathrm{M}(\beta_*)$;

(ii) $K \subseteq \mathrm{M}(\beta_*)$;

(iii) β *is tight*.

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Smoothness From above and Quasi*-Measure

Definition 3.1. Let $C \subseteq I^X$. A function $\xi : C \to I$ is called *smooth from aboveat* $f \in C$ if ξ is monotone and, for any sequence $\{f_n\}_{n=1}^{\infty}$ in C with $f_n \downarrow f$,

$\lim_{n\to\infty} \xi(f_n) = \xi(f)$.

If ξ is smooth from above at each $f \in C$, then ξ is to be called *smooth from above*.

Remark 3.2. A function $\beta : K \to I$ is smooth from above at $f \in K$ if and only if β is monotone and, for any sequence $\{f_n\}_{n=1}^{\infty}$ in K with $f_n \downarrow f$,

$$\beta(f) = \inf\{\beta(g) : g \in K \text{ and } g \geq f_n \text{ for some } n\}.$$

Definition 3.3. We call a family $C \subseteq I^X$ *semicompact* if, every sequence in C having finite meet property (i.e. any finite subcollection of C has nonzero meet) has nonzero meet.

Theorem 3.4. *If β: $K \to I$ is monotone and K is semicompact, then β is smooth from above at* **0.**

Proof. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in K with $f_i \downarrow \mathbf{0}$. Then $\wedge f_i = \mathbf{0}$. Since K is semicompact, $f_n = \mathbf{0}$ for some n, and so $\beta(f_m) = 0$ for $m \geq n$. Thus $\lim_{n\to\infty} \beta(f_n) = 0$, showing that β is smooth at **0**.

Lemma 3.5. *Let $\beta : K \to I$ be smooth from above. Then, for any $f \in K_\delta$, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in K such that $\{f_n\} \downarrow f$. For each such sequence $\beta^*(f) = \inf_n \beta(f_n)$.*

Proof. Let $f \in K_\delta$. Then $f = \bigwedge_{n=1}^{\infty} h_n$ for some sequence $\{h_n\}$ in K, which yields a sequence $\{f_n\}_{n=1}^{\infty}$ in K such that $f_n \downarrow f$.

Let $g \in K$ such that $f \leq g$. Then $\{g \vee f_n\} \downarrow g$ and so $\bigwedge_{n=1}^{\infty} (g \vee f_n) = g$. Since β is smooth from above, we have $\lim_{n\to\infty} \beta(g \vee f_n) = \inf_n \beta(g \vee f_n) = \beta(g)$. Also, since β is monotone, $\beta(f_n) \leq \beta(g \vee f_n)$, for each n, which implies that $\inf \beta(f_n) \leq \beta(g)$. Consequently,

$\inf \beta(f_n) \le \beta^*(f)$. Next, for any $n \in N$, we have $f \le f_n$, and so $\beta^*(f) \le \beta^*(f_n) = \beta(f_n)$. Hence $\beta^*(f) \le \inf_n \beta(f_n)$. Thus $\beta^*(f) = \inf_n \beta(f_n)$.

Definition 3.6. We define $\hat{\beta} : I^X \to I$ by

$\hat{\beta}(f) = \sup\{\beta^*(g) : g \le f, g \in K_\delta\},\ f \in I^X$.

We obtainthe following :

(i) $\hat{\beta}(\mathbf{0}) = 0$ and $\hat{\beta}$ is monotone.

(ii) $\hat{\beta} \le \beta^*$.

(iii) $\hat{\beta} \mid K_\delta = \beta^* \mid K_\delta$; in particular, $\hat{\beta} \mid K = \beta^* \mid K$.

(iv) If β is monotone, then $\hat{\beta} \mid K = \beta$, i.e. $\hat{\beta}$ is an extension of β.

Proposition 3.7. *If* $K = K_\delta$ *then*

(i) $\hat{\beta} \le \beta_*$,

(ii) β *is semifinite* $\Rightarrow \hat{\beta} = \beta_*$.

Proof. (i) Let $f \in I^X$ and $g \in K$ with $g \le f$. Since $\beta^* \mid K \le \beta_* \mid K$ and β_* is monotone, we get $\beta^*(g) \le \beta_*(g) \le \beta_*(f)$. Hence $\hat{\beta}(f) \le \beta_*(f)$.

(ii) For $f \in I^X$, using 2.3 (iv), we get $\hat{\beta}(f) = \beta_*(f)$.

Theorem 3.8. *If* $\beta : K \to I$ *is smooth from above, then* $\beta^* \mid K_\delta$ *is smooth from above*.

Proof. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence in K_δ and $f_n \downarrow f \in K_\delta$. For each n, we obtain a sequence $\{f_{nm}\}_{m=1}^{\infty}$ in K such that $\{f_{nm}\}_{m=1}^{\infty} \downarrow f_n$ and $\beta^*(f_n) = \lim_{m\to\infty} \beta(f_{nm})$. For $n \in N$, set $g_n = f_{1n} \wedge f_{2n} \wedge \ldots \wedge f_{nn}$. Then $g_n \in K$, $\{g_n\}$ is decreasing, $g_n \ge f_n$ for all n, and $g := \lim g_n = \wedge g_n \ge \wedge f_n = f$. Also $f_{kn} \ge g_n$ for $k \le n$. Therefore $f_k = \lim_{n\to\infty} f_{kn} \ge g$. It follows that $f = \wedge f_k \ge g$. Thus $f = g$. We obtain, by Lemma 3.5,

$$\beta^*(f) = \lim_{n\to\infty} \beta(g_n) \ge \lim_{n\to\infty} \beta^*(g_n) \ge \lim_{n\to\infty} \beta^*(f_n).$$

Also, since $f \leq f_n$ for each n, and β^* is monotone, we conclude that $\beta^*(f) \leq \lim_{n\to\infty} \beta^*(f_n)$. Thus $\beta^*(f) = \lim_{n\to\infty} \beta^*(f_n)$, i.e. $\beta^* | K_\delta$ is smooth from above.

Definition 3.9. Let $C \subseteq I^X$ be a lattice. Then $\xi : C \to I$ is called *supermodular* (*submodular* respectively) if, for $f, g \in C, \xi(f)+\xi(g) \leq \xi(f \vee g)+\xi(f \wedge g)$ $(\xi(f)+\xi(g) \geq \xi(f \vee g)+\xi(f \wedge g)$ respectively).

Proposition 3.10. (i) *If β is supermodular, then β_* is supermodular.*

(ii) *If β is submodular, then β^* is submodular.*

Proof. (i) Let $f_1, f_2 \in I^X$. For $\varepsilon > 0$, we obtain $g_1, g_2 \in K$, $f_1 \geq g_1$, $f_2 \geq g_2$ such that $\beta_*(f_1) - \varepsilon/2 < \beta(g_1)$ and $\beta_*(f_2) - \varepsilon/2 < \beta(g_2)$. It follows that

$\beta_*(f_1) + \beta_*(f_2) - \varepsilon < \beta(g_1 \vee g_2) + \beta(g_1 \wedge g_2) \leq \beta_*(f_1 \vee f_2) + \beta_*(f_1 \wedge f_2)$.

Since ε is arbitrary, we get $\beta_*(f_1) + \beta_*(f_2) \leq \beta_*(f_1 \vee f_2) + \beta_*(f_1 \wedge f_2)$.

Proof of (ii) follows analogously.

Theorem 3.11. *Let β:K→I be smooth from above. Then* (1) ⇒ (2) ⇒ (3), *where*

(1) *β is supermodular* ;

(2) *$\beta^* | K_\delta$ is supermodular* ;

(3) (a) *$\hat{\beta}$ is supermodular* ;

(b) *$\hat{\beta}$ is smooth from above*.

Proof. (1) ⇒ (2). Let β be supermodular. Let $f, g \in K_\delta$. Then, by Lemma 3.5, there exist sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ in K such that $f_n \downarrow f$, $g_n \downarrow g$, $\beta^*(f) = \lim_{n\to\infty} \beta(f_n)$ and $\beta^*(g) = \lim_{n\to\infty} \beta(g_n)$. Since $(f_n \wedge g_n) \downarrow (f \wedge g)$, $(f_n \vee g_n) \downarrow f \vee g$ and β is supermodular, we obtain using Lemma 3.5,

$$\beta^*(f) + \beta^*(g) = \lim_{n\to\infty} \beta(f_n) + \lim_{n\to\infty} \beta(g_n) = \beta^*(f \vee g) + \beta^*(f \wedge g).$$

(2) ⇒(3) (a). Let $f, g \in I^X$. For $\varepsilon > 0$, we obtain h_1, h_2 in K_δ such that $h_1 \le f, h_2 \le g$, $\hat{\beta}(f) - \varepsilon/2 < \beta^*(h_1)$ and $\hat{\beta}(g) - \varepsilon/2 < \beta^*(h_2)$. It follows that

$$\hat{\beta}(f) + \hat{\beta}(g) - \varepsilon < \beta^*(h_1) + \beta^*(h_2) \le \hat{\beta}(f \vee g) + \hat{\beta}(f \wedge g).$$

Since ε is arbitrary, we get

$$\hat{\beta}(f) + \hat{\beta}(g) \le \hat{\beta}(f \vee g) + \hat{\beta}(f \wedge g).$$

(2) ⇒ (3) (b). Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence in I^X such that $\{f_n\} \downarrow f$, $f \in I^X$. Since $\hat{\beta}$ is monotone, we have $\hat{\beta}(f_n) \ge \hat{\beta}(f)$, for each n, which yields $\lim_{n\to\infty} \hat{\beta}(f_n) \ge \hat{\beta}(f)$.

We proceed to prove that $\lim_{n\to\infty} \hat{\beta}(f_n) \le \hat{\beta}(f)$. Let $\varepsilon > 0$. We choose $g_n \in K_\delta$ such that $g_n \le f_n$ and

$$\beta^*(g_n) > \hat{\beta}(f_n) - \varepsilon/2^n, \; n = 1, 2, \ldots . \qquad (3.11.1)$$

Put $h_n = g_1 \wedge g_2 \wedge \ldots \wedge g_n$. Then $h_n \in K_\delta$ and $\{h_n\} \downarrow h \in K_\delta$. Now, by (3.11.1), we get $\beta^*(h_1) = \beta^*(g_1) > \hat{\beta}(f_1) - \varepsilon/2$. Since $\beta^* | K_\delta$ is supermodular, using (3.11.1),

$$\beta^*(h_2) = \beta^*(g_1 \wedge g_2) = \hat{\beta}(f_2) - (\varepsilon/2 + \varepsilon/2^2).$$

Suppose that $\beta^*(h_m) \ge \hat{\beta}(f_m) - \sum_{i=1}^{m} \frac{\varepsilon}{2^i}$. Using supermodularity of β^* on K_δ and (3.11.1), we obtain

$$\beta^*(h_{m+1}) \ge \beta^*(h_m \wedge g_{m+1}) \ge \beta^*(h_m) + \beta^*(g_{m+1}) - \beta^*(h_m \vee g_{m+1})$$

$$> \hat{\beta}(f_m) - \sum_{i=1}^{m} \frac{\varepsilon}{2^i} + \hat{\beta}(f_{m+1}) - \frac{\varepsilon}{2^{m+1}} - \hat{\beta}(f_m \vee f_{m+1})$$

$$= \hat{\beta}(f_{m+1}) - \sum_{i=1}^{m+1} \frac{\varepsilon}{2^i}.$$

Thus, by induction, we deduce that

$$\beta^*(h_n) \ge \hat{\beta}(f_n) - \sum_{i=1}^{n} \frac{\varepsilon}{2^i}, \quad \text{for all } n. \qquad (3.11.2)$$

Note that, since $g_n \le f_n$, for all n, $h = \overset{\infty}{\underset{n=1}{\wedge}} h_n = \overset{\infty}{\underset{n=1}{\wedge}} g_n \le \overset{\infty}{\underset{n=1}{\wedge}} f_n = f$. Also, by Theorem 3.8, $\beta^* | K_\delta$ is smooth from above. Hence, by (3.11.2), we obtain

$$\hat{\beta}(f) \ge \beta^*(h) = \lim_{n\to\infty} \beta^*(h_n) \ge \lim \hat{\beta}(f_n) - \varepsilon,$$

and therefore $\hat{\beta}(f) \ge \lim_{n\to\infty} \hat{\beta}(f_n)$. Thus $\hat{\beta}$ is smooth from above.

Definition 3.12. Let $C \subseteq I^X$. We call a function $\rho : I^X \to I$ *C-inner regular* if, for each $f \in I^X$,

$$\rho(f) = \sup\{\rho(g) : g \le f, g \in \mathrm{C}\}.$$

Proposition 3.13. The function $\hat{\beta}$ *is* K_δ*-inner regular*.

Proof. Since $\beta^* | K_\delta = \hat{\beta} | K_\delta$, the result follows.

Definition 3.14. We call a supermodular, smooth from above function $\xi : I^X \to I$ a *quasi*-measure* on X if $\xi(\mathbf{0}) = 0$, and call the pair (X, ξ), a *quasi*-measure space*.

Theorem 3.15. *Every supermodular, smooth from above function defined on a lattice K in* I^X *containing* **0** *and* **1**, *can be extended to a* K_δ *-inner regular quasi*-measure on* I^X.

Proof. Follows from Definition 3.6 (i), Theorem 3.11 and Proposition 3.13.

References

[**1**] W. Adamski, Tight set functions and essential measure, in: D. Kölzow and D. Maharam-Stone(Ed.), Lecture Notes inMath.**945**, Springer-Verlag, 1981, 1-14.

[**2**] J.R. Choksi, On compact contents, *J. London Math. Soc.***33** (1958), 387-398.

[**3**] P.R. Halmos, *Measure Theory*, D. Van Nostrand Company, Inc., New York, 1950.

[**4**] J.L. Kelley, M.K. Nayak and T.P. Srinivasan, *Pre-measures on Lattices of Sets* II Sympos. *On Vector Measures*, Salt Lake City Utah, 1972.

[**5**] J.L. Kelley and T.P. Srinivasan, Premeasures on lattice of sets, *Math. Ann.***190** (1971), 233-241.

[**6**] J. Kisynski, On the generation of tight measures, *Studia Math.***30** (1968), 141-151.

[**7**] Mona Khare and Bhawna Singh, Weakly tight functions and their decomposition, *Int. Jour. of Math. and Math Sc.*, **48**(2)(1975),2991-2998

[**8**] P. Morales, Extension of a tight set function with values in a uniform semigroup, in: D. Kölzow and D. Maharam-Stone(Ed.), Lecture Notes in Math.**945** Springer-Verlag, 1981, 282-290.

[**9**] M.K. Nayak and T.P. Srinivasan, Scalar and vector valued premeasures, *Proc. Amer. Math. Soc.***48** (2) (1975), 391-396.

[**10**] B.J. Pettis, On the extension of measures, *Ann. of Math.***54** (1) (1951), 186-197.

[**11**] A.S. Sastry and K.P.R. Sastry, Measure extensions of set functions over lattices of sets, *J. Indian Math. Soc.***41** (1977), 317-330.

[**12**] F. Topsϕe, Compactness in spaces of measures, *Studia Mathematica*, **36** (1970), 195-212.